

SUM RULES FOR RADIATIVE QUARKONIUM DECAYS FROM SHORT-RANGE INTERACTIONS

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Sum rules previously derived for proton decay are extended and applied to treat effects of bound state spectroscopy on radiative quarkonium decays. The transition involves boson (photon or pion) emission followed by quark annihilation. Sum rules for the contributions from different intermediate bound states are derived by using closure and the assumptions: 1) that the boson emission is described by a plane wave or multipole operator which satisfies a wave equation, 2) that the annihilation depends on the bound state wave function or its derivative at the origin.

Many interesting decay processes involve the emission of a boson (e.g. a photon or a pion) by an initial single-hadron state and subsequent annihilation of its quark constituents (quark-antiquark pair or three quarks) via a short-range interaction proportional to the wave function at the origin for the bound state [1-5]. There can also be mixing of states in the quarkonium spectrum via a short-range interaction; e.g. a hyperfine

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interaction or annihilation and subsequent pair creation, associated with decays of the above type [2,3]. The intermediate single-hadron states which are annihilated or created by the short-range interaction can include the whole spectrum of bound states; e.g. higher radial excitations. Many results can depend upon the parameters of these higher states which are generally not known. This paper examines some general sum rules which give results essentially independent of these parameters.

Sum rules have been used in proton decay to estimate the contributions of higher resonances to pole diagrams [1]. The same formalism can be generalized to apply to all such processes involving boson emission and quark annihilation, e.g. radiative quarkonium decays. The matrix element for a transition from an initial state $|i\rangle$ to a final state $|fB\rangle$, where B denotes the emitted boson, can be written

$$\langle fB|M|i\rangle = K \sum_X \frac{X(0)\langle X|B(q)|i\rangle}{[E_X - E_i + E_B]} \quad (1a)$$

where K is a constant depending upon the decay process but independent of hadron wave functions, X denotes an intermediate hadron state produced after the emission of a boson with momentum q, $X(0)$ is the wave function at the origin of hadron X, $B(q)$ is the factor in the operator describing the boson emission which depends upon the variables of the hadron bound state (the remaining factor is absorbed in the constant K), and E_X , E_i and E_B are the energies of the state X, the initial state and the emitted boson respectively.

A more general form is useful to include decays in which the annihilation occurs from a state of nonzero orbital angular momentum and depends upon the derivatives of the wave function at the origin

$$\langle fB|M|i\rangle = K \sum_X \frac{\langle r=0;q|A|X\rangle\langle X|B(q)|i\rangle}{[E_X - E_i + E_B]} \quad (1b)$$

where A denotes an operator describing the annihilation process and $|r=0;q\rangle$ denotes a state of total momentum q in which the quark and antiquark are at the same point and normalized so that $\langle r=0;q|X\rangle = X(0)$. For the case where the annihilation involves the wave function at the origin, A is simply the identity operator and the expression (1b) reduces to eq. (1a). For electric dipole transitions, where the annihilation involves the derivative of the wave function at the origin, the operator $A=ip$ and ip acts as a derivative on the wave function $|X\rangle$ in eq. (1b).

We now note that two sum rules can be obtained from eqs. (1a) and (1b) by closure if the energy denominators are replaced by some approximate or average energy and taken outside the summation.

The Matrix Element Sum Rule:

$$S_0 = \sum_X \langle r=0;q|A|X\rangle\langle X|B(q)|i\rangle = \langle r=0;q|AB(q)|i\rangle \quad (2a)$$

The Energy Weighted Sum Rule:

$$S_1 = \sum_X \langle r=0;q|A|X\rangle\langle X|B(q)|i\rangle [E_X - E_i] = \langle r=0;q|A[H, B(q)]|i\rangle \quad (2b)$$

where H is the Hamiltonian of the bound state including the center-of-mass motion so that recoil energies are automatically included. In many cases of practical interest like electromagnetic multipole transitions and pion emission the operator B(q) depends only upon the coordinates and spins of the constituents and not on momenta, while the Hamiltonian depends upon momenta only in the kinetic energy term. Thus

$$[H, B(q)] = \left(\frac{1}{4M}\right) [P^2, B(q)] + \left(\frac{1}{M}\right) [p^2, B(q)] \quad (3a)$$

where M is the mass of the constituents, P is the center-of-mass momentum and p is the relative momentum. The relevant factor in $B(q)$ describing center-of-mass motion is a plane wave with wave number q . For the case where the relative motion $B(q)$ is described either by a plane wave with wave number $q/2$ or by any term in the multipole expansion of such a plane wave, we can use the fact that $B(q)$ satisfies the wave equation and replace its Laplacian by $-q^2 B(q)$ and $(-q/2)^2 B(q)$ in evaluating the commutator for the center-of-mass and relative motions respectively to obtain

$$[H, B(q)] |i\rangle = \left(\frac{2i}{M}\right) \vec{\nabla} B(q; r) \cdot \vec{p} |i\rangle + \left(\frac{1}{2M}\right) q^2 B(q; r) |i\rangle \quad (3b)$$

where $\vec{\nabla}$ refers to the relative motion only, and the analogous term for center-of-mass motion vanishes because the initial state $|i\rangle$ is chosen to have total momentum zero. The sum rule (2b) then becomes

$$S_1 = \frac{q^2}{2M} S_0 - \frac{2i}{M} \langle r=0 | A \vec{\nabla} B \cdot \vec{p} | i \rangle \quad (4)$$

These sum rules can be used to obtain approximate expressions or bounds for the transition matrix element (1) by using the following identities to expand the energy denominator

$$\frac{1}{E_X - E_i + E_B} = \frac{1}{E_i + E_B - E_i} - \frac{E_X - E_i}{(E_i + E_B - E_i)(E_X + E_B - E_i)} \quad (5a)$$

$$\frac{1}{E_X - E_1 + E_B} = \frac{E_1 + E_2 + E_B - 2E_1}{(E_1 + E_B - E_1)(E_2 + E_B - E_1)} - \frac{E_X - E_1}{(E_1 + E_B - E_1)(E_2 + E_B - E_1)} + \frac{(E_X - E_1)(E_X - E_2)}{(E_1 + E_B - E_1)(E_2 + E_B - E_1)(E_X + E_B - E_1)} \quad (5b)$$

where E_1 and E_2 are any two energies; e.g. the energies of the two lowest intermediate states. The purpose of these expansions is seen by noting that the first term on the right hand side of eq. (5a) and the first two terms in (5b) give expressions easily evaluated by the sum rules (2) when they are substituted into eq. (2), while the last term is constructed to cancel the contributions from the dominant intermediate states. Thus, substituting eqs. (2) and (5) into eq. (1b) gives

$$\langle fB|M|i \rangle = K \frac{S_0}{E_1 + E_B - E_1} \left[1 - \frac{1}{S_0} \sum_X \frac{(E_X - E_1)}{(E_X + E_B - E_1)} \langle r=0; q | A | X \rangle \langle X | B(q) | i \rangle \right] \quad (6a)$$

$$\langle fB|M|i \rangle = K S_0 \left[\frac{E_1 + E_2 + E_B - 2E_1}{(E_1 + E_B - E_1)(E_2 + E_B - E_1)} \right] \left[1 - \frac{S_1/S_0}{E_1 + E_2 + E_B - 2E_1} + \frac{1}{S_0} \sum_X \frac{(E_X - E_1)(E_X - E_2) \langle r=0; q | A | X \rangle \langle X | B(q) | i \rangle}{(E_1 + E_2 + E_B - 2E_1)(E_X + E_B - E_1)} \right] \quad (6b)$$

Note that neglecting the remainder terms in eq. (6) is a much better approximation than neglecting the contributions from the higher intermediate states in the original expression (1). The contributions of the leading terms in the expansions (6a) are reduced by factors $\left(\frac{E_X - E_1}{E_1 + E_B - E_1} \right)$ in eq. (6a) and $\left(\frac{E_X - E_1}{E_1 + E_B - E_1} \right) \left(\frac{E_X - E_2}{E_2 + E_B - E_1} \right)$ in eq. (6b) relative to their contributions in (1). These are considerable reduction factors for any state X whose excitation energy $E_X - E_1$ is much less than E_B .

The simplest application of this sum rule is to cases described by eq. (1a), like proton decay [1] or radiative quarkonium decays to pseudoscalar

final states [4,5]. In these cases $A=1$, the second term on the right hand side of eq. (4) vanishes and the operator $B(q)$ does not depend upon momenta. The two sum rules (2a) and (4) then simplify to give

$$S_0 (A=1) = \langle f | B(q; r=0) | i \rangle_s i(0) \quad (7a)$$

$$S_1 (A=1) = \frac{q^2}{2M} S_0 \quad (7b)$$

where $B(q; r=0)$ denotes the operator B when $r=0$ is substituted, and the subscript s on the matrix element denotes that the matrix element is calculated only in spin space; the spatial part has already been removed by the factor $i(0)$, the wave function at the origin of the initial state $|i\rangle$.

The right hand side of the expression (7b) is just the energy of a particle of mass M recoiling with momentum q . This "free recoil energy" recalls similar sum rules for the Mössbauer effect [6], where a photon is also emitted from a bound system and the question arises of how the bound state spectrum affects the radiative transition.

The analogy with the Mössbauer effect points out the existence of three mass or energy scales in the problem, the quark mass M , the boson momentum q and the characteristic excitation energies $(E_2 - E_1)$ of the bound state spectrum. "Strong binding" occurs when the free recoil energy is smaller than the bound state energies. The binding effects then become important and the transitions are dominated by the low-lying excited states. This in fact occurs for the case of the radiative decay of the upsilon at 9460 MeV to a state called the zeta [4,5,7,8] at 8300 MeV. Then $q \sim 1$ GeV, $2M \sim 10$ GeV, and the energy spacings are ~ 0.5 GeV which is much larger than the free recoil energy of 0.1 GeV.

In the case of proton decay, a simple relation has been noted [1] between the sum rule (7a) and the pole approximation in which only the contribution of the initial state (2) is included in the calculation of the matrix element (1a)

$$\langle fB|M|i \rangle_{\text{pole}} = \frac{K \langle i|B(q)|i \rangle i(0)}{M_i - M_f} = \frac{K}{M_i - M_f} \frac{\langle i|B(q)|i \rangle}{\langle i|B(q); r=0|i \rangle} S_0 \quad (8a)$$

where M_f is the mass of the final state remaining after the emission of the boson. This has been compared with the fictitious degenerate case in which all intermediate states $|X\rangle$ are degenerate with the initial state $|i\rangle$,

$$\langle fB|M|i \rangle_{\text{deg}} = \frac{KS_0}{M_i - M_f} \quad (8b)$$

The pole contribution (8a) includes only the contribution from the lowest state. The degenerate model includes all contributions, but overestimates the higher contributions. The two differ only by the factor $\langle i|B(q)|i \rangle / \langle i|B(q); r=0|i \rangle$ which is just the form factor of the initial state. This form factor is thus a measure of the error in using only the pole approximation.

An approximate formula can be obtained from eq. (6a) with the state $|1\rangle$ taken as the lowest intermediate state above the initial state $|i\rangle$. This gives the result obtained for the proton decay case in ref. 1

$$\begin{aligned} \langle fB|M|i \rangle &= \frac{KS_0}{M_i - M_f} + \frac{(M_i - M_1)}{(M_i - M_f)} \langle fB|M|i \rangle_{\text{pole}} - \dots \\ &= \langle fB|M|i \rangle_{\text{pole}} + \frac{KS_0 - (M_i - M_1) \langle fB|M|i \rangle_{\text{pole}}}{M_i - M_f} - \dots \end{aligned} \quad (8c)$$

For the case where the terms in the summation in (6b) are positive definite and give a negative contribution to the matrix element, the expression (8c) also gives an upper bound for the matrix element as well as an approximate correction to the pole approximation.

We now apply the sum rules (7) and the relation (6b) to the case of radiative upsilin decay to a pseudoscalar final state [4,5,7]. We neglect hyperfine splittings and set E_1 and E_2 in (6b) to be the energies of the 1s and 2s states (T and T') of the upsilin system. We then obtain

$$\langle P_Y | M | T \rangle = \frac{KS_0}{q} \left[1 - \frac{q}{2M} + \dots \right] \quad (9)$$

where T denotes either the 1s or 2s states.

This result (9) is expected to be a good approximation in the region where q is small compared with $2M$ but lower than the excitation energies $E_x - E_1$ of the dominant intermediate states. The first correction term in (9) is of order q/M , the dominant higher terms in the sum (6b) contain factors of order $\frac{E_x - E_1}{q}$. Similar results have been obtained by explicit calculations using a potential model [4,5].

In the pseudoscalar case the operator $B(q)$ describes a magnetic dipole transition and a factor of q appears in $B(q)$ which exactly cancels the factor q in the denominator of eq. (9). The right hand side thus depends on the properties of the initial state $|i\rangle$ mainly via the factor $i(0)$ in S_0 and otherwise only very weakly through the value of q in the second term. This suggests that the transition rate, normalized to the wave function at the origin, is about the same for the ground and first excited states of the upsilin family.

For the case of decay to a scalar meson final state the intermediate states are produced via an electric dipole transition, and the situation is somewhat more complicated. One phenomenon which can occur is a suppression of the transition by destructive interference between the contributions from two different intermediate states.

The radiative decay of the T' to a scalar can proceed either via the lowest 0^+ state of the bottomonium system or via the first radially excited state. These two contributions can be shown to have opposite phase by the following general and simple argument [2]. Let us choose a phase convention in which all wave functions are positive at or near the origin. The factors involving wave functions at the origin are then always positive by convention and have the same phase for all intermediate states. The electric dipole matrix element can be expected to have a positive phase for the transition between the two radially excited states, since both wave functions have one node and will have a positive relative phase at both short and long distances where they contribute positively to the overlap integral. There can only be a very small negative contribution if the two nodes do not occur exactly at the same radius.

The transition from the T' to the lowest 0^+ state involves a radial overlap integral between a wave function with one node and a wave function with no nodes. The phase of this integral depends upon whether the integral is dominated by short distances, where the relative phase of the two functions is positive or by long distances where the relative phase is negative. Since the electric dipole operator involves a factor r which enhances the long-range contribution, we can expect the sign of the overlap integral to be determined by the long-range phase which is negative. Thus, the contributions from the two dominant scalar intermediate states to radiative decays of the T' have

opposite phase and a suppression by destructive interference can occur.

In the case of T' decay, one would expect the dominant matrix element to be the one to the radially excited 0^+ state which has the same number of nodes. The matrix element to the lowest 0^+ state should be smaller and interfere destructively. But the contribution of this lowest state to the transition amplitude is enhanced by the smaller energy denominator. The degree of destructive interference thus depends crucially on the relative enhancement of the lower state due to the difference in energy denominators. One can see that there must be a value of the final state mass where an exact cancellation occurs, since the contribution from the lowest 0^+ state can be made arbitrarily large by choosing B in eq. (1) to make the energy denominator for this lowest state arbitrarily small. Whether the parameters in a realistic case are such that an appreciable cancellation can occur requires a wave detailed investigation, such as the explicit calculations of refs. [4,5].

We now show how the two sum rules can be used to show the strong cancellations due to interference. The pseudoscalar intermediate states are p -wave quark-antiquark states whose wave functions vanish at the origin. The transition matrix elements for this case involve the derivative of the wave function at the origin.

We therefore take $A = i\vec{p}$ in the sum rules (2a) and (4) and the operator $\vec{B}(q)$ as the electric dipole operator which satisfies the wave equation and vanishes at the origin. Let us normalize $\vec{B}(q)$ so that $\vec{B}(q)$ behaves like the vector \vec{r} near the origin. Then

$$\langle r=0; q | AB(q) | 1 \rangle = \langle r=0; q | [A, B(q)] | 1 \rangle = \langle r=0; q | \text{div } \vec{B}(q) | 1 \rangle = 3i(0) \quad (10a)$$

Combining the commutator (5b) and eq. (10a) then gives

$$\begin{aligned} \langle r=0; q | A[H, B(q)] | i \rangle &= \left(\frac{2i}{M}\right) \langle r=0; q | A \vec{\nabla} B(q) \cdot \vec{p} | i \rangle + \left(\frac{1}{2M}\right) \langle r=0; q | A q^2 B(q) | i \rangle = \\ &= \left(\frac{2}{M}\right) \langle r=0; q | p^2 | i \rangle + \left(\frac{3}{2M}\right) q^2 i(0) \end{aligned} \quad (10b)$$

Note that in the simple approximation for the electric dipole operator as just the relative coordinate \vec{r} , only the first term of eq. (10b) appears [4]. The second term can be considered as the correction which appears when the full dipole operator which satisfies the wave equation is used. Substituting these relations into the sum rules (2a) and (4) gives

$$S_0 = 3i(0) \quad (11a)$$

$$S_1 = 2[E_1 - V(0) + \left(\frac{3}{4M}\right) q^2] i(0) \quad (11b)$$

where we replace the kinetic energy term p^2/M by $E - V(r)$. Note that only the relative motion appears here as $|i\rangle$ has total momentum zero, and it is the reduced mass $(\frac{M}{2})$ that appears in the kinetic energy.

The energy weighted sum rule (11b) is not useful as it stands because it depends upon the value of the potential $V(0)$ at the origin, and most relevant potentials like the Coulomb potential are singular at the origin. However, this term can be eliminated by taking the difference of the sum rules for two different initial states; e.g. the 1S and 2S.

$$\frac{S_1(i)}{i(0)} - \frac{S_1(j)}{j(0)} = 2(E_i - E_j) + \left(\frac{3}{2M}\right) [q(i)^2 - q(j)^2] \quad (12)$$

This result can now be substituted into eq. (6b). We note that the energy of the final meson, $E_f = E_1 - E_B$ can be taken as independent of the initial state $|i\rangle$ by neglecting the small change in recoil energy with $|i\rangle$, thus

$$\begin{aligned} \frac{\langle fB|M|i\rangle}{K_i(0)} - \frac{\langle fB|M|j\rangle}{K_j(0)} &= \\ &= \frac{1}{(E_1 - E_f)(E_2 - E_f)} \left[3(E_j - E_1) - \frac{S_1(1)}{i(0)} + \frac{S_1(j)}{j(0)} + \dots \right] \\ &= \frac{1}{(E_1 - E_f)(E_2 - E_f)} \left[5(E_j - E_1) (1+F) + \dots \right] \end{aligned} \quad (13a)$$

where

$$F = \frac{3}{10M} \frac{q(1)^2 - q(j)^2}{E_1 - E_j} \quad (13b)$$

defines the correction to the simple approximation of the dipole operator. Since for the T-T' system $q(i) - q(j)$ is equal to $E_1 - E_j$ if we neglect recoil corrections and E_1 is approximately $2M$, the correction F is of order $(\frac{3}{10})[q(i) + q(j)]/M$ or $(\frac{3}{5})[E_1 + E_j - 2E_f]/E_1$.

We now apply this relation to the case of the T and T' decays and note that the correction terms have contributions only from the third and higher transitions. We next substitute the sum rule (11a) into eq. (6a) for the T initial state, where it is reasonable to expect the lowest transition to be dominant

$$\frac{\langle f\gamma|M|T\rangle}{KT(0)} = \frac{3}{M_{1p} - M_f} \left[1 - \frac{1}{3} \sum_X \frac{M_X - M_{1p}}{M_X - M_f} \frac{\langle r=0; q | A | X \rangle \langle X | B(q) | T \rangle}{T(0)} \right] \quad (14a)$$

Combining eqs. (13a) and (14a) then gives

$$\frac{\langle f\gamma | M | T' \rangle / T'(0)}{\langle f\gamma | M | T \rangle / T(0)} = 1 - \frac{5}{3} \frac{(M_{T'} - M_T)}{(M_{2p} - M_f)} [1 + F + \dots] \quad (14b)$$

Eqs. (14) already show the qualitative features discussed above.

The opposite signs of the two dominant contributions to the T' decay on the right hand side of eq. (14b) indicate the presence of destructive interference and the possibility of a cancellation.

The right hand side of eq. (14b) vanishes when

$$M_f = M_{2p} - [5(1+F)/3][M_{T'} - M_T] + \dots \quad (15)$$

The amplitude for the T' decay thus vanishes when the mass of the final state mass M_f satisfies the condition (15). Substituting the values for the upsilon system, $M_T = 9460$ MeV, $M_{T'} = 10025$ MeV, $M_{1p} = 9873$ MeV, $M_{2p} = 10233$ MeV, neglecting the higher order term and setting $F=1$ as a first approximation gives $M_f = 9294$ MeV for the mass at which the T' amplitude vanishes. This is very close to the value obtained by an explicit calculation for the particular potential used to fit the upsilon spectrum [4,5].

When M_f is set equal to the zeta mass, 8320 MeV, the right hand side of eq. (14) is 0.51. This is consistent with results obtained by the explicit calculations. When the correction factor F is included, the result changes to 0.44 and gives 9239 MeV for the mass at which the T' amplitude vanishes.

Note that for this case the leading correction term neglected in eq. (14a) is multiplied by the factor $\frac{M_{2p} - M_{1p}}{M_{1p} - M_\zeta} = 0.2$ in comparison with its contribution to the direct calculation (1b). The leading correction term neglected in eq. (13a) comes from the $3p$ intermediate state and is multiplied

by $\frac{(M_{3p}-M_{1p})(M_{3p}-M_{2p})}{(M_{1p}-M_{\zeta})(M_{2p}-M_{\zeta})} = 0.09$, where we have taken $M_{3p} = 10600$ MeV. Thus these expansions are expected to converge much more rapidly than the expansion (1b).

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